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# Optimal Income Taxation and Hidden Borrowing and Lending: The First-Order Approach in Two Periods\*

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## Abstract

We provide sufficient conditions for the validity of the first-order approach for two period dynamic moral hazard problems, where the agent can save and borrow secretly. We show that in addition to the concavity requirements for the standard moral hazard problem, non-increasing absolute risk aversion (NIARA) utility functions and Frisch elasticity of leisure less than one imply that the agent's problem is jointly concave in effort and asset decisions when facing the optimal contract. We also characterize the optimal contract in detail. One of the key observation is that the possibility of hidden asset accumulation makes the supporting tax-transfer system more regressive (or the optimal compensation scheme more convex) under a general class of preferences (HARA).

*Keywords:* Moral Hazard, Hidden Savings, First Order Approach, Optimal Income Taxation.

*JEL:* C61, D82, E21, H21.

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# 1 Introduction

This paper has two targets. First, we provide sufficient conditions for the validity of the first-order approach (FOA) for two period dynamic moral hazard problems where the agent can save and borrow secretly, and we characterize the optimal contract. Second, we study the implication for optimal income taxation of hidden/anonymous access to the credit market (or the availability of a storage technology).

Recently, dynamic principal-agent models became very popular instruments to study several diverse issues such as design of optimal social insurance schemes (e.g. unemployment insurance, and disability insurance), bank-firm financing relationships, efficient compensation contracts, and optimal capital taxation. Most of these models assume that the agent's consumption-savings decision is observable (and fully contractable) by the principal. However, it is also well-known that this assumption is potentially very dangerous, because if the agent is given a hidden (or not contractable) opportunity to save then he would deviate from the optimal contract by saving (and possibly exerting less effort) (Rogerson, 1985a). Therefore, the possibility of hidden asset accumulation will lead to a different optimal contract. This problem is also relevant empirically, as in most of the above-mentioned applications, the contractability and observability of asset accumulation cannot be guaranteed.

The FOA consists in replacing the incentive compatibility constraints of the agent by the corresponding first-order necessary conditions from the agent's decision problem. Since the seminal works by Mirrlees (1971) and Holmström (1979) it became obvious that the study of the moral hazard models is much easier if one can rely on the first-order condition approach (FOA). The simplification of the incentive compatibility constraint becomes even more important in a dynamic environment when the principal faces an additional information problem because the agent has secret access to the credit market. Rogerson (1985b) and Jewitt (1988) provide conditions for the validity of the FOA for the static principal-agent model. Their strategy is to show that when facing the optimal contract, the agent's problem is concave hence the first order conditions are actually not only necessary but also sufficient for the optimality of the agent's decisions.

It is not known under what conditions the FOA can be applied to multi-period principal-agent problems with hidden asset accumulation. In fact, Kocherlakota (2004) finds cases (linearity of both the effort cost and the effort's impact on output) where - although the standard conditions that guarantee the validity of the first-order approach in the static model are verified - the agent's problem is not concave when he is allowed to enter the credit market. Intuitively, the non-concavity

is a consequence of potential benefits from *jointly* decreasing effort and increasing savings. The necessary first-order conditions may not capture these second-order gains. In this paper, we provide sufficient conditions for the two-period model under which the agent's problem is concave when facing the optimal contract and therefore the incentive compatibility constraint for the agent's decisions can be replaced by its necessary and sufficient first-order conditions. In particular, we show that within the family of non-increasing absolute risk aversion (NIARA) utility functions in consumption, a strong concavity condition on the distribution function together with the 'spanning condition with dominance' proposed by Grossman and Hart (1983) guarantees that the FOA is applicable. We also show that the concavity condition we impose on the probability distribution is essentially equivalent to a requirement on the utility function of leisure such that the Frisch elasticity of leisure is less than unity. Our conditions imply that most of the (additively separable) utility functions and many effort specifications used in applications allow for a first order condition representation of the problem. Further, empirical studies seem to confirm both the NIARA and the Frisch elasticity condition.

We provide two main characterization results. First, we show that as opposed to hidden information moral hazard models (see Allen, 1985 and Cole and Kocherlakota, 2001) self-insurance is not optimal in this environment. This result is general in the sense that it does not even require the validity of the FOA. Second, we show that, similarly to the pure moral hazard case with observable assets, under the standard monotone likelihood ratio condition, consumption is monotone increasing in income. We also study extensively the curvature of the optimal consumption allocation as a function of income. This is particularly important, because the curvature determines the progressivity of the supporting tax-transfer system. Our main conclusion is that, under preferences of the linear risk tolerance class (HARA), the possibility of the hidden asset accumulation makes (*certeris paribus*) the optimal consumption a more convex function of income, hence under hidden asset accumulation the *optimal tax system becomes more regressive* (or less progressive) compared to the case where asset accumulation is observable. Intuitively, the principal would like to discourage the agent from using hidden savings and exerting smaller effort. To discourage savings, not only consumption has to be more backloaded (second period consumption needs to increase) compared to the case when savings are observable. We find that whenever the coefficient of absolute risk aversion is decreasing and convex (HARA utility functions), the optimal distribution of these consumption increases across income levels makes consumption more convex. To the best of our knowledge, this paper constitutes the first attempt to study the implication of hidden savings for

optimal income taxation.<sup>1</sup>

In addition to allowing a sharp characterization of the optimal contract, finding conditions for the validity of the first order approach is important for at least two other reasons. First, as explained in Ábrahám and Pavoni (2008), Werning (2001, and 2002), and Kocherlakota (2004), the first order approach is crucial for being able to write the problem in a tractable recursive form. Second, it can be shown that whenever the first order approach is valid the optimal tax on asset holdings takes a simple form. In particular, imposing linear taxes on savings which are uniform across ex-post shocks is optimal.<sup>2</sup>

Unfortunately, our analytical results for the two period model cannot be easily extended to a framework with more than two periods. In Ábrahám and Pavoni (2008), we propose a recursive reformulation of the multi-period (or infinite horizon) problem based upon the first-order condition representation and verify the validity of the first-order condition approach ex post, numerically. This paper constitutes a first step toward the analytical study of this class of dynamic moral hazard models, which already provides important insights about the problem. In fact, virtually all existing papers that study the moral hazard problem with hidden asset trade or storage analytically, either use very special closed form solutions or use two period models.<sup>3</sup>

To the best of our knowledge, there is only one other paper which studies systematically the issue of the validity of the first-order approach in this class of models. Williams (2006) gives

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<sup>1</sup>In contrast, there is a relatively large literature that uses this class of models to study the implications for optimal capital income taxation (e.g., Golosov et al., 2003; Kocherlakota, 2005; Albanesi and Sleet, 2006; Golosov and Tsyvinski, 2007; and Gottardi and Pavoni, 2008)

<sup>2</sup>This observation is almost implicit in Kocherlakota (2005), and it has been shown to be true for a wide set of assets by Gottardi and Pavoni (2007). In the latter, it is also shown that this simple tax system implies a *positive* (expected) tax on capital.

<sup>3</sup>In a two period principal agent relationship, Bizer and DeMarzo (1999) show that hidden access to the credit market reduces total welfare with respect to the no asset market case. They focus on the possibility of increasing welfare by allowing the entrepreneur to default on the debt. Bisin and Rampini (2006) study the effect of bankruptcy provision, in a two period model similar to that of Bizer and DeMarzo, where agents have hidden access to *insurance contracts* and can default on the principal insurer as well. In addition to no-default, we do not allow agents to secretly trade assets other than a risk free bond. Chiappori et al. (1994) and more recently Park (2004) analyze the optimal contract with discrete effort. They find that - under some conditions - a renegotiation-proof contract always implements the minimum level of effort. We consider a continuous-effort model, where the planner can commit not to renegotiate the contract ex post. Kocherlakota (2004) characterizes the optimal UI transfer scheme in an infinite horizon two-output moral hazard model with hidden savings, where agents' preferences are linear in effort, and effort affects linearly job-finding probabilities. Werning (2002) solves analytically a similar two-output model with multiplicative separable CARA utility. We consider a two period model which allows for both a general class of preferences and a much more general production technology.

sufficient conditions for the validity of the FOA for a large set of *continuous time* principal-agents models. There are few important differences between our approach and that of Williams that make the two papers complementary to each other. First, although his conditions are stated for a very large set of models and for any time horizon, they are not satisfied in a context where there is a linear return/storage technology for assets such as assumed here. Second, Williams focuses on the dynamic aspects of the problem and - as in most of the literature in continuous time models - considers a stochastic production technology with normally distributed shocks (the Brownian model). We focus on the two period problem with shocks on a bounded support but allow for virtually any distribution function and any pattern for the likelihood ratios over the support.<sup>4</sup>

The next section presents the model and derive minimal conditions for optimality. Then we introduce the first-order condition approach and provide conditions for the concavity of the agent's problem in the optimum in Section 3. In Section 4, we study the curvature of consumption and its implications for optimal income taxation in presence of hidden savings. Further analysis of the optimal contract and a discussion about possible extensions is provided in Section 5. Section 6 concludes.

## 2 Model

Consider a relationship between a risk neutral principal/planner and a risk averse agent, that lasts for two periods:  $t = 0, 1$ . The model builds on the typical dynamic moral hazard problem and assumes that consumption occurs at the beginning of each period (together with the effort decision).<sup>5</sup>

**Preferences** The agent derives utility from consumption  $c_t \geq \underline{c} \geq -\infty$  and effort  $e_t \geq 0$  according to:  $u(c_t) - v(e_t)$ , where both  $u$  and  $v$  are strictly increasing and twice continuously differentiable functions, and  $u$  is strictly concave whereas  $v$  is convex. We normalize  $v(0) = 0$ . The agent's discount factor will be denoted by  $\beta \geq 0$ .

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<sup>4</sup>Schattler and Jaryoung (1997) discuss how removing the assumption of Brownian motion in the standard principal-agent model in continuous time changes the properties of the optimal contract. Schattler and Jaryoung (1997) also confirm the finding in Mirrlees (1975) and show that for any 'discretization' of the continuous time model, the problem with normally distributed shocks is not well defined. In particular, the optimal contract approximates arbitrarily well the first best allocation by imposing extreme punishments and rewards upon events with very small probability.

<sup>5</sup>This timing is very common, for example, in the optimal unemployment insurance literature (e.g., Hopenhayn and Nicolini, 1997).

**Production and endowments** At date  $t = 0$ , the agent has a *fixed* endowment  $y_0$ . At date  $t = 1$ , there are  $N$  possible output levels  $Y := \{y_1, \dots, y_N\}$  with  $y_i < y_{i+1}$ . The realization  $y_i \in Y$  is publicly observable, while the probability distribution over  $Y$  is affected by the agent's unobservable effort level  $e_0$  that is exerted at  $t = 0$ . The conditional probabilities are defined by the smooth functions:<sup>6</sup>  $p_i(e_0) := \Pr\{y = y_i \mid e_0\}$ . As in most of the optimal contracting literature, we assume *full support*, that is  $p_i(e_0) > 0$  for all  $i = 1, \dots, N$ , and all  $e_0$ . There is no production or any other action at  $t \geq 2$ .

**Markets** At each date, the agent can buy or (short)-sell a risk-free bond  $b_t$  which costs  $q$  consumption units today and pays one unit of consumption tomorrow. The agent has no access to any other insurance market other than that delivered by the principal (exclusivity). We assume that asset decisions and consumption levels are private information to the agent.

Given the structure of the problem, the agent will never be able to borrow at  $t = 1$  hence  $b_1 \geq 0$ . Monotonicity of preferences will guarantee that the agent does not want to leave any positive amount of assets at date 1 either. So  $b_1 = 0$  for all states  $i$ . Similarly, since  $v$  is strictly increasing,  $e_1 = 0$  at all  $i$ .

**Contracts** A *contract*  $\mathcal{W} := (\boldsymbol{\tau}, \boldsymbol{\sigma})$  is constituted by a transfer scheme  $\boldsymbol{\tau} := (\tau_0, \{\tau_i\}_{i=1}^N)$  where  $\tau_0$  and  $\tau_i$  represent the transfers the individual receives in period  $t = 0$  and in period  $t = 1$  conditional on realization  $y_i$ , respectively. To simplify the analysis, we separate the principal's transfer plan from the components of the allocation under the agent's control, which are  $\boldsymbol{\sigma} := (e_0, b_0)$ . Given  $\mathcal{W}$ , the agent's utility is

$$U(e_0, b_0; \boldsymbol{\tau}) := u(y_0 + \tau_0 - qb_0) - v(e_0) + \beta \sum_{i=1}^N p_i(e_0) [u(y_i + \tau_i + b_0)]. \quad (1)$$

Recall that a key assumption in our model is that the planner cannot observe how the agent allocates his income  $y_0 + \tau_0$  between consumption  $c_0$  and asset accumulation  $qb_0$ . As usual, to guarantee solvency of the agent for every contingency, we impose the 'natural' borrowing limit:  $b_0 \geq \underline{c} - \min_{i=1, \dots, N} \{y_i + \tau_i\}$ .<sup>7</sup>

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<sup>6</sup>In particular, we require the function  $\mathbf{p} : E \rightarrow \Delta^N$  to be continuous and continuously differentiable. Here,  $\Delta^N := \{x \in \mathbb{R}^N \mid x \geq 0 \text{ and } \sum_i x_i = 1\}$ .

<sup>7</sup>The enforceability of the repayment of debt obtained through anonymous credit lines is an important and delicate issue, which is common to many environments and that we do not address here. With minor modifications to the analysis, one could just impose  $b_0 \geq 0$ , which could be enforced in an anonymous credit market. In this case, asset accumulation could also be interpreted as a private storage technology. At the end of the paper, we will consider the case with liquidity constraints more in detail.

The risk neutral planner faces the same credit market as the agent, therefore her discount rate is  $q$ . Her preferences/profits are

$$V(e_0, b_0; \boldsymbol{\tau}) := -\tau_0 + q \sum_{i=1}^N p_i(e_0) (-\tau_i). \quad (2)$$

**Efficiency** An *optimal contract* is the contract that maximizes the principal's discounted expected profits, that is<sup>8</sup>

$$V(U_0) := \max_{\mathcal{W}} V(e_0, b_0; \boldsymbol{\tau}), \quad (3)$$

subject to the *participation constraint*

$$U(e_0, b_0; \boldsymbol{\tau}) \geq U_0, \quad (4)$$

and the *incentive compatibility constraint*

$$(b_0, e_0) \in \arg \left\{ \max_{e, b} U(e, b; \boldsymbol{\tau}) \text{ s.t. } e \geq 0, y_0 + \tau_0 - \underline{c} \geq qb \geq -q \min_i \{y_i + \tau_i - \underline{c}\} \right\}. \quad (5)$$

We will denote this problem as **(P)**. In order to make the problem of some interest, we assume that  $U_0 > -\infty$ .

Note, that there is indeterminacy in the contract between  $\tau_0$  and  $b_0$ . The planner can implement the same allocation to the agent with a contract  $\{\tau_0, \{\tau_i\}_{i=1}^N, e_0, b_0\}$  and with the contract  $\{\tau_0 - \varepsilon, \{\tau_i + \varepsilon/q\}_{i=1}^N, e_0, b_0 - \varepsilon/q\}$ . In other words, since the planner and the agent face the same return in the credit market there are a continuum of optimal contracts. In this paper, without a loss of generality, we will study the one specific optimal contract which implements  $b_0 = 0$ . Because of these observations, we will sometimes refer to the combination of  $e_0$  and  $c_i := y_i + \tau_i$ ,  $i = 0, 1, 2, \dots, N$  as a contract.

We can interpret this setup in several different ways. First, this framework may describe a private insurance relationship, where  $y_0$  is the agent's (verifiable) initial net wealth and he can affect future outcomes by his action  $e_0$ . The insurance company is the principal, who offers a contract, which will imply an initial fee ( $\tau_0$ ) and a (net) insurance payment ( $\tau_i$ ) dependent on the realized state. In this case, if we set  $U_0$  such a way, that  $V(U_0) = 0$ , the insurance contract will deliver zero profits to the insurance company. When designing the conditions of the contract, the insurer has to take into account that the agent can influence the likelihood of the different events. In addition, in our environment, the insurer cannot observe the agent's consumption either, therefore

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<sup>8</sup>Existence can be shown, for example, by a simple extension to Grossman and Hart (1983) by assuming an upper bound on  $e_0$  and unbounded below  $u$ .



the optimal insurance contract has to take into account that the agent is able to transfer resources intertemporally through the credit market.

Another interpretation is a two-period compensation contract, where  $y_i$  is the surplus which can be divided by a risk-neutral owner/shareholder and a risk-averse worker/manager. In this case,  $y_i + \tau_i$  is the wage of the worker in a given state and date, while  $V(U_0)$  is the profit of the owner. Here again, the wages have to provide the right incentives for the agent to exert high effort. Moreover, high punishments (low wages) for low second period surpluses may not be incentive compatible because the worker can save against them at date  $t = 0$ .

Alternatively, and this is our preferred application, we can interpret this relationship as an optimal tax/transfer provision problem (social insurance), where the principal is a benevolent social planner whose objective is to maximize the welfare of the citizens. The (small open) economy contains a continuum of ex ante identical agents who face the above contract and can influence their date 1 income realizations by working hard or shirk. The planner offers a tax/transfer system to insure them against idiosyncratic risk and, at the same time, provide them appropriate incentives for working hard. In this case, setting  $U_0$  such a way that  $V(U_0) = 0$  would be the socially optimal allocation. Since, for this application, (3) can be interpreted as the social planner's dynamic budget constraint, by the law of large numbers  $V(U_0) = 0$  is equivalent to an (intertemporal) balanced budget requirement.

The main implications of this framework are known when hidden borrowing and lending are not allowed. We will briefly review them below. However, Rogerson (1985a) has shown that in this setup, the agent is left with incentives to save in the optimal contract. That implies that the optimal allocation of this model is different from the ones where asset accumulation is observable and contractable or not allowed. Moreover, in all of the three above examples private (non-observable, non-contractable or non-taxable) savings are empirically relevant. Neither insurance companies nor shareholders/owners can control the agent's consumption saving decision, however their wealth level affects the effectiveness of the incentive scheme. Finally, typically governments cannot have a full control over agents' consumption saving decisions either, because agents can keep their savings in low-interest (and not observable) instruments such as local and foreign currency or they can have access to foreign accounts.

## 2.1 Preliminary Characterization and Self-insurance

Under very general conditions (in particular without requiring that the first order approach is valid) one can prove that transfers  $\tau_i$  cannot uniformly increase with  $y_i$ .<sup>9</sup>

**Proposition 1 (Minimal Insurance)** If  $u$  is unbounded below or above, and  $\underline{c} > -\infty$ , then  $\tau_i$  cannot be monotone increasing (not even weakly increasing) with  $y_i$  for all  $i$ .

The intuition is as follows. A transfer scheme that has the property of  $\tau_i$  increasing for all  $i$  cannot be optimal since the planner could twist the scheme so that to decrease  $\tau_i$  in states with large  $y_i$  and increase it in bad states (states with low  $y_i$ ). Since the agent is risk averse and the planner is risk neutral, we expect the principal to gain by absorbing some of the income risk the agent faces. In particular, the principal should be able to provide more insurance to the agent than he can achieve without the principal's participation.

Proposition 1 has a somewhat important consequence. Allen (1985) and Cole and Kocherlakota (2001) study the effect of hidden asset accumulation in a hidden information moral hazard model. They find that the constrained efficient allocation does not differ from that in a 'pure bond economy,' i.e., the allocation the agents could obtain by insuring themselves through borrowing and lending, without the planner's provision of additional insurance. In terms of our optimal taxation examples, this result would imply that the planner have no role in enhancing welfare as the only incentive compatible allocation will provide no additional insurance to the agents compared to a pure bond economy.

Note however that a pure bond economy would correspond to a situation of constant  $\tau_i = \tau$  for  $i = 1, 2, \dots, N$ . Proposition 1 hence shows that this is cannot be the case for our 'hidden action' moral hazard model.

**Corollary to Proposition 1 (Impossibility of Self-Insurance)** Under the assumptions in Proposition 1, the optimal contract never delivers the self insurance allocation.

Since the contract  $\tau_i \equiv \tau$  is available to the principal, an equivalent way of stating the result is as follows. If we let  $\underline{U}_0$  be the utility value obtainable by the agent under self-insurance, we have  $V(\underline{U}_0) > 0$ . What is then the key difference between the hidden information and the hidden action moral hazard models that generates this contrasting implications? As it is clear from our line of proof, the full support assumption plays a major role in the result. Under this condition, the

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<sup>9</sup>All the the proofs not shown in the main text can be found in the Appendix.

agent has incomplete control over the realizations  $y_i$ . Hence, the planner can implement schemes that impose a tax payment in some states and a transfer in others in such a way that the agent is not able to avoid paying the taxes with certainty. Clearly, this result extends to any multi-period setting.<sup>10</sup>

Unfortunately, one cannot characterize the optimal contract much further analytically without the use of the first-order approach. In the next section we will introduce this approach and provide sufficient conditions for its validity. In Section 4, we then use the first order approach to characterize the optimal contract in greater detail.

### 3 The First-Order Approach

It is not difficult to see that condition (5) describes a complicated set of constraints: it is equivalent to a bidimensional continuum of inequalities. The first order approach replaces the incentive constraint (5) for the corresponding stationary points of the agent's maximization problem with respect to  $e_0$  and  $b_0$ . This strategy brings the number of inequality constraints down to only two.

In what follows, we will *assume interiority of the optimal contract*, that is we assume that the original problem (**P**) has a solution consumption levels strictly above 0 in both dates and for all states and that  $e_0 > 0$ . It is easy to see that the moral hazard problem is not interesting if the optimal effort is the minimal effort.<sup>11</sup> We will follow Rogerson (1985b) and replace the incentive constraint with the following necessary conditions for optimality of the decisions for  $e_0$  and  $b_0 (= 0)$  :

$$\begin{aligned} U_e(e_0, 0; \tau) &\geq 0 \\ &\text{and} \\ U_b(e_0, 0; \tau) &\leq 0. \end{aligned} \tag{6}$$

The intuition for the above conditions is simple. There are two key restrictions we impose on the

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<sup>10</sup>Moreover, consistently with the previous intuition, in AP we show that, in fact, the full-support assumption is not required. It suffices to exclude the possibility of distributions over  $Y$  which are degenerate at one state  $i$  for some  $e$ .

<sup>11</sup>Interiority may be required for a more technical reason. As emphasized by Mirrlees (1975), when the solution to the original problem is at the corner then the first order approach might fail to deliver even necessary conditions for an optimum. In our case, we could guarantee interiority in effort by assuming that  $e$  is taken within an open set as in Jewitt (1988). Alternatively, we could avoid the complications due to the corner solution by assuming that  $v'(0) = 0$ . In this case, it is very easy to see that for  $e_0 = 0$  both the optimal contract and the solution obtained using the FOA will deliver full insurance. Interiority with respect to consumption can be guaranteed by imposing that consumption is chosen within an open interval (as in Grossman and Hart, 1983), or by requiring that  $\lim_{c \rightarrow \underline{c}} u'(c) = -\infty$ .

contract. First, we guarantee that the agent is not willing to reduce his effort level or shirk. The second condition is the usual inequality version of the Euler equation: we require the agent not willing to save.

Notice, that both above constraints depend on the agent's equilibrium choices of consumption and effort alone. Given our normalization on  $b_0 = 0$ , it will indeed be convenient to describe the principal agent relationship as one where the principal decides directly the consumption level of the agent at each state. We define  $\mathbf{c} := \{c_i\}_{i=0}^N$  and rewrite the planner problem as

$$\max_{e_0, \mathbf{c}} V(e_0, 0; \mathbf{c})$$

subject to  $c_i \geq 0$ ,  $e_0 \in E$ , the relaxed incentive constraint (6), which can be written as follows:

$$v'(e_0) + \beta \sum_i p'_i(e_0) u(c_i) \geq 0 \quad (7)$$

$$qu'(c_0) - \beta \sum_i p_i(e_0) u'(c_i) \geq 0, \quad (8)$$

where  $c_i := y_i + \tau_i$ ; and the participation constraint (4), or

$$U(e_0, 0; \mathbf{c}) \geq U_0.$$

The expressions  $V(e_0, 0; \mathbf{c})$  and  $U(e_0, 0; \mathbf{c})$  have the obvious meaning.<sup>12</sup> This problem will be denoted by **(R)**. We refer to this second problem as *relaxed*, because the set of contracts that satisfy the constraints of **(R)** contains the set of contracts that satisfy the constraints of the original problem **(P)**. The remaining of this section is devoted to provide conditions under which the solution to the relaxed problem **(R)** is identical to the solution of the original (unrelaxed) problem **(P)** given by equations (3) to (5).

Until now, there were not known conditions for discrete time models under which the above first-order conditions are necessary and sufficient for incentive compatibility. In Ábrahám and Pavoni (2008), we approach this issue by a numerical ex post verification procedure for generic multi-period

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<sup>12</sup>They are the analogous to  $V(e_0, 0; \boldsymbol{\tau})$  and  $U(e_0, 0; \boldsymbol{\tau})$ . For example, we have

$$U(e_0, 0; \mathbf{c}) := u(c_0) - v(e_0) + \beta \sum_i p_i(e_0) u(c_i).$$

With this notation, the constraints in (6) can also be written as

$$\begin{aligned} U_e(e_0, 0; \mathbf{c}) &\geq 0, \\ U_b(e_0, 0; \mathbf{c}) &\leq 0. \end{aligned}$$

problems. However, that procedure requires a numerical solution of the problem. There, we also consider the case  $u(c, e) = -\exp\{-\rho c + g(e)\}$ , with  $g$  increasing and convex. These preferences are not additive separable and therefore we cannot directly apply our approach, however it is easy to see that the first order approach is valid under even less restrictive assumptions in this case, and that the validity of the FOA in this case extends to any multiperiod setting. For further details, we demand the reader to Ábrahám and Pavoni (2008).

Williams (2006) provides sufficient conditions for concavity for a class of continuous time models with and without hidden savings. The key distinction between his approach and the one followed here, is that he provides sufficient conditions for the cases where either the price of the bond  $q$  is not constant but a convex function of  $b$ , or  $q$  is constant but the agent's utility directly depends on wealth in a strictly concave way.

We will find sufficient conditions for concavity in two steps. First, we will characterize the 'relaxed' optimal contract  $(\mathbf{R})$  by imposing the agent's first-order conditions instead of (5). Then, we prove that, under appropriate conditions, date 1 consumption ( $c_i$ ) of the agent is changing monotonically with his income ( $y_i$ ). Finally, we will show that under somewhat stricter conditions, monotonicity of consumption implies that the agent's problem is concave in the relaxed optimal contract. This will imply that under this set of stricter conditions our use of the first-order condition approach is actually justified as the solution to  $(\mathbf{R})$  also constitutes a solution to  $(\mathbf{P})$ .

### 3.1 Monotonicity of Consumption

We start by analyzing the properties of consumption based on the relaxed problem  $(\mathbf{R})$ . We need to first introduce some well known properties of the probability shifting functions  $\mathbf{p}$  and the utility function, which we will use extensively later. We will make use of the following assumptions.

**NIARA.** The utility function  $u$  exhibits *non-increasing absolute risk aversion*, that is, the ratio  $-\frac{u''(c)}{u'(c)} := a(c_i)$  is non-increasing in  $c$ .

**MLR.** The probability shifting functions  $\mathbf{p}$  has the *monotone likelihood ratio* property, that is, for each  $e \geq 0$  the ratio  $\frac{p'_i(e)}{p_i(e)}$  is non-decreasing in  $i$ .

We can now state our second characterization result.

**Proposition 2 (Monotonicity)** Assume NIARA and let  $(\mathbf{c}, e_0)$  be a solution to  $(\mathbf{R})$ . (i) Either  $c_i = \underline{c}$  or  $c_i$  moves together with the likelihood ratio  $\frac{p'_i(e_0)}{p_i(e_0)}$ . (ii) Under MLR,  $c_i$  increases with  $i$  for  $i = 1, 2, \dots, N$ .

**Proof.** Consider problem **(R)**, and denote by  $\mu$ ,  $\xi$  and  $\lambda$  the Kuhn-Tucker multipliers associated to the constraints (7), (8), and (4) respectively. By standard conditions they are all nonnegative. The necessary conditions for optimality (with respect to  $c_0$  and  $c_i$ ) are:

$$\frac{1}{u'(c_0)} \geq \lambda + \xi \frac{qu''(c_0)}{u'(c_0)} \text{ with equality if } c_0 > 0; \quad (9)$$

Moreover, for  $i = 1, \dots, N$  we either have  $c_i = 0$ , or:

$$\frac{q}{\beta u'(c_i)} = \lambda + \mu \frac{p'_i(e_0)}{p_i(e_0)} - \xi \frac{u''(c_i)}{u'(c_i)} = \lambda + \mu \frac{p'_i(e_0)}{p_i(e_0)} + \xi a(c_i). \quad (10)$$

Note that from (10) and  $\mu \geq 0$ , the expression  $\frac{q}{\beta u'(c_i)} - a(c_i)\xi$  must move together with the likelihood ratio  $\frac{p'_i(e_0)}{p_i(e_0)}$ . By concavity of  $u$  and NIARA, both  $\frac{1}{u'}$  and  $-a(c_i) = \frac{u''}{u'}$  increase with  $c_i$ . Since  $\xi \geq 0$  consumption must move with  $i$  in the same direction as  $\frac{p'_i(e_0)}{p_i(e_0)}$ . **Q.E.D.**

The previous proposition replicates a standard result in the contract theory literature with no access to the credit market.<sup>13</sup> In terms of our second example, this result says that the wages are increasing with the observed output of the agent. More generally, the first part of Proposition 2 emphasizes that in our environment consumption varies across states in proportion to  $\frac{p'_i(e_0)}{p_i(e_0)}$  *alone*. That is, consumption only responds to the informational content of the outcome realization  $y_i$  on the effort level  $e_0$ . We will discuss the implications of these results in Section 4 in more detail. Note that this contrasts the self-insurance allocation where - since taxes  $\tau$  are constant across states -  $c_i$  moves one to one with  $y_i$  regardless of the informational content of income levels. All the above properties are strict as long as both  $\mu$  and  $\xi$  are strictly positive. The next Lemma establishes this latter fact when  $e_0 > 0$ .

**Lemma 1** Assume NIARA and let  $(\mathbf{c}, e_0)$  be a solution to **(R)**. Then (i) If  $c_i > \underline{c}$  for  $i = 1, 2, \dots, N$  then  $U_e = U_b = 0$ , that is, constraints (7) and (8) are both satisfied with equality. (ii) If both  $c_i > \underline{c}$  for all  $i$  and  $e_0 > 0$ , then both  $\mu > 0$  and  $\xi > 0$ . (iii) If  $u$  satisfies the Inada:  $\lim_{c \rightarrow \underline{c}} u'(c) = \infty$ , then (i) holds under NIARA without the interiority requirement on consumption.

Of course, if  $u$  is unbounded below or  $\underline{c} = -\infty$  the interiority condition  $c_i > \underline{c}$  can be guaranteed a priori. Lemma 1 shows that whenever we are using the first order approach, under mild regularity conditions we can in fact (without loss of generality) impose (6) with equality. If in addition we are looking for interior solutions for effort then the multipliers associated to the incentive constraints are

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<sup>13</sup>Note that the properties of the optimal contract in the case when the agent have no access to the credit market can be recovered from the above optimality conditions assuming  $\xi = 0$ .

positive. The following remark shows that the multiplier associated to the participation constraint is positive whenever we have an interior solution for  $c_0$ .

**Remark** It is easy to see that whenever  $c_0 > 0$  we have  $\lambda > 0$ . This is so since whenever  $U(e_0, 0; \mathbf{c}) > U_0$  the principal could always reduce  $c_0$ . This modification to the contract does not affect the effort incentive constraint, relaxes the Euler equation, and increases the principal's profits.

We now move to the issue of concavity of the agent's problem.

### 3.2 Sufficient Conditions for Global Concavity

Lemma 1 guarantees that condition (6) in **(R)** is satisfied with equality. Whenever the solution to problem **(P)** is interior, a necessary condition for incentive compatibility is to have (6) satisfied with equality. Since **(R)** solves a relaxed problem, the value associated to it cannot be lower than that associated to **(P)**. Therefore, if we show that the contract solving **(R)** is actually incentive compatible, then we have in fact derived the optimal contract. We hence say that the *first order approach is justified* whenever a solution to the relaxed problem **(R)** delivers a correct solution to the original problem **(P)**. If the agent's problem is globally jointly concave in  $b$  and  $e$  when facing the optimal transfer scheme  $\boldsymbol{\tau}$ , the use of the first-order approach is justified as the first order conditions of the agent's problem become sufficient conditions for incentive compatibility. Recall, that given  $\boldsymbol{\tau}$ , the utility of the agent for each combination  $(e, b)$  is

$$U(e, b; \boldsymbol{\tau}) := u(y_0 + \tau_0 - qb) - v(e) + \beta \sum_i p_i(e) u(y_i + \tau_i + b). \quad (11)$$

Global concavity is obviously guaranteed if the associated Hessian matrix is negative definite for all  $e$  and  $b$ ; where the Hessian is given by

$$H := \begin{bmatrix} -v''(e) + \beta \sum_{i=1}^N p_i''(e) u(\hat{c}_i) & \beta \sum_i p_i'(e) u'(\hat{c}_i) \\ \beta \sum_i p_i'(e) u'(\hat{c}_i) & q^2 u''(\hat{c}_0) + \beta \sum_i p_i(e) u''(\hat{c}_i) \end{bmatrix}. \quad (12)$$

In the previous expression, we have denoted  $\hat{c}_0 := y_0 + \tau_0 - qb$ , and  $\hat{c}_i := y_i + \tau_i + b$  for  $i = 1, 2, \dots, N$ . This notation emphasizes that the consumption levels in  $H$  can differ from those delivered by the optimal contract (which are denoted simply as  $c_i$ ). Note importantly, that since the bond pays equal amounts in each state  $i > 0$ , if in the optimal contract  $c_i$  change monotonically with  $i$  then  $\hat{c}_i$  will keep this property. In fact, for all  $i$  and  $j$ , we have that  $\hat{c}_i - \hat{c}_j = c_i - c_j$ .

By the concavity of the utility function  $u$ , it is easy to see that problem is concave in  $b$  alone. Rogerson (1985b) shows that if the distribution function is concave (CDF)<sup>14</sup> the problem is also concave in  $e$  alone, whenever the optimal consumption scheme is monotone in output (which is guaranteed by MLR and NIARA). Unfortunately, the CDF and MLR conditions will not guarantee the concavity of the agent's problem *jointly* in  $(e, b)$ ; that will require a stronger assumption. Joint (strict) concavity requires that the Hessian is negative definite. This is not true in general, not even under CDF and MLR.<sup>15</sup> The previous discussion however guarantees that, under NIARA, CDF and MLR, both entries in the main diagonal of  $H$  are negative.  $H$  is hence negative definite if and only if  $\det H > 0$ .<sup>16</sup> Now, observe the off-diagonal elements:  $\beta \sum_i p'_i(e) u'(\hat{c}_i)$ . They measure the gains from joint deviations of postponing consumption and reducing effort and vice versa (saving and shirking). Intuitively, the problem is concave if the gains from these joint deviations are sufficiently small compared to the loss induced by moving away from the optimal levels of  $e_0$  and  $b_0$  in the main diagonal. Our assumptions so far are able to deliver the following result:

**Lemma 2** (i) If  $u$  satisfies NIARA, we have  $\sum_i p'_i(e) u'(\hat{c}_i) \leq 0$ . (ii) Under NIARA, MLR and CDF, we have  $\sum_{i=1}^N p''_i(e) u(\hat{c}_i) < 0$  and

$$\det H > \left( \sum_{i=1}^N p''_i(e) u(\hat{c}_i) \right) \left( \sum_i p_i(e) u''(\hat{c}_i) \right) - \left( \sum_i p'_i(e) u'(\hat{c}_i) \right)^2.$$

The last line of the lemma indicates that - since  $u$  is concave - we can focus on the second period effects of the deviations. In order to be able to provide simple sufficient conditions for  $\det H > 0$ , we now impose further structure on the probability shifting functions. We assume that there exists an *increasing* function  $0 \leq \Gamma(e) \leq 1$  such that

$$p_i(e) = \Gamma(e) \pi_{ih} + (1 - \Gamma(e)) \pi_{il} \quad (13)$$

with  $\sum_i \pi_{ik} = 1$  for  $k = h, l$  and with  $\frac{\pi_{ih}}{\pi_{il}}$  *non-decreasing* in  $i$ . This requirement is called the 'spanning condition with dominance', and it has been used in the literature in order to guarantee

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<sup>14</sup>The definition is standard. The functions  $\{p_i(e)\}_{i=1}^N$  satisfy the CDF condition if  $F'_I(e)$  is non-negative for every  $e$  and  $I \leq N$ , where  $F_I(e) = \sum_{i=1}^I p_i(e) = 1 - \sum_{i=I+1}^N p_i(e)$ .

<sup>15</sup>Kocherlakota (2004), provides an additive-separable 'linear-linear' counter-example where he assumes that  $p'_i(e) = v''(e) = 0$ , and therefore

$$\det H = - \left( \sum_i p'_i(e) u'(\hat{c}_i) \right)^2 < 0.$$

<sup>16</sup>Obviously, if the latter inequality is weak  $H$  is negative *semi*-definite.



monotonicity of the consumption scheme in the moral hazard model with no hidden asset accumulation (see Grossman and Hart, 1983; and Atkeson, 1991). The intuition is simple: we have two base distributions given by the vectors  $\pi_l$  and  $\pi_h$ , in such a way that the distribution given by  $\pi_h$  first-order stochastically dominates  $\pi_l$ . Intuitively, by exerting higher effort the agents shifts the probability distribution of future outcomes towards the ‘better’ distribution. Note that probability shifting functions given in (13) will satisfy the MLR and the CDF condition as long as  $\Gamma$  is increasing and concave. Therefore, we know from our previous results that, if  $u$  is NIARA and  $\Gamma$  is increasing and concave, then consumption is monotone and the agent’s problem is concave in  $e_0$  and  $b_0$  individually. The following main result will establish a sufficient condition for joint concavity.

**Proposition 3 (FOA)** Assume  $u$  is NIARA, and that the increasing function  $\Gamma(e)$  in (13) is such that  $f_\Gamma(e) := \frac{(\Gamma'(e))^2}{-\Gamma''(e)(1-\Gamma(e))} \leq 1$  for all  $e \in E$ . Let  $(\tau)$  be a transfer scheme solving **(R)**. Then the agent’s problem is concave when facing  $(\tau)$ . As a consequence, the first order approach is justified: A contract  $(e_0, \mathbf{c})$  solves **(R)** if and only if it solves the original problem **(P)**.

**Proof.** First, for  $i = 1, 2, \dots, N$  let us denote  $\Delta\pi_i := \pi_{ih} - \pi_{il}$ ,  $u_i := u(\hat{c}_i)$ ,  $u'_i := u'(\hat{c}_i)$  and  $u''_i := u''(\hat{c}_i)$ . Also notice that (13) implies that  $p'(e) = \Gamma'(e)\Delta\pi_i$  and  $p''(e) = \Gamma''(e)\Delta\pi_i$ . Then, the determinant of the Hessian is given by the following expression:

$$\det H = \left( v''(e) + \beta\Gamma''(e) \sum_i \Delta\pi_i u_i \right) \left( q^2 u''(\hat{c}_0) + \beta \mathbf{E}[u''_i] \right) - \left( \beta\Gamma'(e) \sum_i \Delta\pi_i u'_i \right)^2.$$

Statement (ii) in Lemma 2 can be written as  $\det H > (\Gamma''(e) \sum_i \Delta\pi_i u_i) (\mathbf{E}[u''_i]) - (\Gamma'(e) \sum_i \Delta\pi_i u'_i)^2$ . After writing  $\mathbf{E}[u''_i]$  as  $(\sum_i \pi_{ih} u''_i - (1 - \Gamma(e)) \sum_i \Delta\pi_i u''_i)$  and getting rid of the term  $\sum_i \pi_{ih} u''_i < 0$ , we obtain<sup>17</sup>

$$\det H > \Gamma''(e) \sum_i \Delta\pi_i u_i \left( -(1 - \Gamma(e)) \sum_i \Delta\pi_i u''_i \right) - \left( \Gamma'(e) \sum_i \Delta\pi_i u'_i \right)^2. \quad (14)$$

Now, we rearrange the expression in (14) as follows:

$$\frac{\det H}{-\Gamma''(e)(1 - \Gamma(e)) \left( \sum_i \Delta\pi_i u'_i \right)^2} > \frac{\sum_i \Delta\pi_i u_i \sum_i \Delta\pi_i u''_i}{\left( \sum_i \Delta\pi_i u'_i \right)^2} - \frac{(\Gamma'(e))^2}{-\Gamma''(e)(1 - \Gamma(e))}.$$

The last term of the previous expression is precisely  $f_\Gamma$ . Hence, the assumption  $f_\Gamma(e) \leq 1$  implies that  $\det H$  is positive whenever  $R_u := \frac{\sum_i \Delta\pi_i u_i \sum_i \Delta\pi_i u''_i}{\left( \sum_i \Delta\pi_i u'_i \right)^2} \geq 1$ . Now, we need to only prove that

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<sup>17</sup>Note that we are entitled to make the simplification since  $\Gamma''(e) < 0$  and  $\Delta\pi_i$  and  $u_i$  are perfectly aligned since both  $\Delta\pi_i$  and  $\hat{c}_i$  increases with  $i$ , hence  $\Gamma''(e) \sum_i \Delta\pi_i u_i < 0$ .

NIARA utility functions will have the property  $R_u \geq 1$ . Note that  $\sum_i \Delta\pi_i = 0$  and our previous results imply that  $\Delta\pi_i$  and  $\hat{c}_i$  both increase with  $i$ .

**Lemma 3** If  $u$  exhibits the NIARA property then  $\frac{\sum_i \alpha_i u_i \sum_i \alpha_i u_i''}{(\sum_i \alpha_i u_i')^2} \geq 1$  for every vector of weights  $\alpha$  such that  $\sum_i \alpha_i = 0$  and both  $u_i$  and  $\alpha_i$  are either both increasing or both decreasing in  $i$  for all  $i$ . Conversely, if  $u$  is concave and such that  $R_u \geq 1$  for all vectors of weights  $\alpha$  with  $\sum_i \alpha_i = 0$  and such that  $u_i$  and  $\alpha_i$  are either both increasing or both decreasing in  $i$  for all  $i$ , then  $u$  is NIARA.

**Q.E.D.**

In order to understand better the condition  $f_\Gamma(e) \leq 1$ , note that by concavity and strict monotonicity of  $\Gamma$ ,  $f_\Gamma(e) > 0$  for all effort levels. On the other hand, a linear  $\Gamma$  would clearly violate this requirement because there  $f_\Gamma(e) = \infty$ . Therefore, our restriction requires that there is enough concavity in the probability shifting function. It should be also clear from our analysis that the concavity condition on the function  $\Gamma$  can be relaxed by imposing some strong convexity conditions on  $v(e)$ . When  $v$  is strongly convex, one can make appropriate change in variables and relax the requirements on  $\Gamma$ . More formally, the first order approach is justified for any couple of functions  $\hat{\Gamma}$  and  $\hat{v}$  (defined on a set  $H$ ), such that the change in variable  $e := \hat{v}(h)$  and  $\Gamma(e) := \hat{\Gamma}(\hat{v}^{-1}(e))$  leads to a function  $\Gamma$  defined on the set  $E$  that has the required properties: it is increasing and  $f_\Gamma(e) \leq 1$ .

Following this same line of reasoning, we can explain our condition on  $\Gamma$  in terms of the cost function  $v$ . Suppose we have a function  $\Gamma$  that satisfied our condition and  $v$  is linear. If we make the change in units so that to obtain a *linear*  $\hat{\Gamma} : \hat{\Gamma}(h) = h := \Gamma(e)$ , hence  $\hat{v}(h) := \Gamma^{-1}(h)$ , we can ask what are the properties on  $\hat{v}$  which are implied by the condition  $f_\Gamma \leq 1$ . Given a linear relationship between  $h$  and the (expected) earnings we can interpret it as labor supply. We can hence write the agent's problem using an increasing concave utility function of leisure given by  $-\hat{v}(h) := g(1-h) = -\Gamma^{-1}(h)$ .<sup>18</sup> Taking first and second derivatives of these identities, we obtain

$$g'(1-h) = \frac{1}{\Gamma'(e)} \quad \text{and} \quad g''(1-h) = \frac{\Gamma''(e)}{(\Gamma'(e))^3}.$$

Combining these equations and using the definition of  $f_\Gamma(e)$  we get

$$-\frac{g''(1-h)(1-h)}{g'(1-h)} = \frac{1}{f_\Gamma(e)} \geq 1.$$

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<sup>18</sup>Note that the linearity of  $\hat{\Gamma}$  implies that  $h \in [0, 1]$  in this case.

In other terms,  $f_\Gamma(e)$  represents the (intertemporal) elasticity of leisure. One of the most widely used specifications for  $g$  is unit elasticity one:  $g(1-h) = B \ln(\phi(1-h))$ , which obviously implies  $f_\Gamma(e) = 1$ . The unit elasticity is particularly interesting since it can be obtained by inverting the expression  $1 - \Gamma(e) = k \exp(-\rho e)$ , where  $\rho = \frac{1}{B}$  and  $k = \frac{1}{\phi}$ . These results might be quite useful for applied work. First, typically applied researchers choose utilities of consumption from the NIARA family and either use  $\Gamma(e) = 1 - k \exp(-\rho e)$  with some convex effort cost function, or they use a linear function for  $\Gamma$  and a constant elasticity function for leisure. Our results suggest that for these cases, the first-order condition approach tend to be valid under hidden savings as well, so the optimal allocations can be characterized in greater detail. Second, empirical evidence suggests that both of our conditions are likely to be satisfied in the data. Virtually all empirical estimates of labor supply elasticity for men find it between 0 and 0.5. Heckman and MaCurdy (1980) find higher estimates for women, but their estimates are still below one. One of the most recent structural estimations is done by Domeij and Floden (2006) who find values for the Frisch elasticity of labor supply between .3 and .56. To our best knowledge, all estimations for  $u$  reveal NIARA, for example Guiso *et al.* (2001) find decreasing and convex relative risk aversion.

Finally, notice that the result stated in Lemma 3 has its own independent interest as it provides a new characterizations of class of concave and NIARA utility functions. CARA utilities are examples of utility function satisfying the condition as we have  $R_u = 1$ , regardless of the risk aversion parameter. In order to get a more intuition about this result, assume that  $N = 2$ , hence  $y_2 > y_1$  and  $\Delta\pi_1 = \alpha = -\Delta\pi_2$ . Then

$$R_u = \frac{\Delta u \Delta u''}{(\Delta u')^2},$$

where  $\Delta u^{(n)} = u^{(n)}(c_2) - u^{(n)}(c_1)$ . If we divide both the numerator and the denominator by  $(c_2 - c_1)^2 > 0$  and take the limit for  $c_2 \rightarrow c_1$ , we get  $\frac{u' u'''}{(u'')^2}$ . That is  $R_u \geq 1$  is just a difference counterpart of  $u' u''' \geq (u'')^2$  which is a defining property of NIARA utility functions. Therefore, intuitively, our condition expresses the requirement of non-increasing absolute risk aversion in difference terms instead of differential terms.

## 4 Regressive Optimal Taxation

So far, we established two characterization results. Transfers cannot increasing with output for all  $i$ , while consumption typically does (at least under MLR). We have emphasized that these characteristics of the cross sectional consumption distribution are common to both the standard moral hazard model and the model with hidden assets we consider in this paper. In this section,

we characterize the curvature of the optimal consumption scheme.

Clearly, the curvature of  $\mathbf{c}$  is closely related to the progressivity of the tax/transfer system. Recall again, that we can fix  $b_0 = 0$  without loss of generality. We say that the transfer scheme  $\tau$  is *progressive* (*regressive*) if  $\frac{c_{i+1}-c_i}{y_{i+1}-y_i}$  is decreasing (increasing) in  $i$ . As expected, the self-insurance allocation will have  $\frac{c_{i+1}-c_i}{y_{i+1}-y_i} = 1$  for all  $i$ . This definition implies that whenever consumption is a convex (concave) function of income we have a regressive (progressive) tax system supporting it. In terms of our first period taxes and transfers  $\tau_i$ , in a progressive tax system taxes ( $\tau_i < 0$ ) are increasing faster than income does. At the same time, for the states when the agents is receiving a transfer ( $\tau_i > 0$ ), transfers are increasing slower than income decreases. The opposite happens when we have a regressive tax-transfer scheme. If the scheme is progressive, incentives are provided more by imposing “large penalties” for low income levels, since consumption is decreasing more for low income/output levels. On the other hand, if the scheme is regressive, then incentives are provided by larger rewards for high output realizations because consumption is increasing more for high output realizations. If the scheme is proportional these rewards and punishments are balanced. The next proposition provides sufficient conditions for the progressivity and regressivity of the optimal transfer scheme.

**Proposition 6** Assume the FOA is justified, that the optimal contract is interior and that the likelihood ratio is monotone and convex<sup>19</sup> (concave) and that  $\frac{1}{u'(c)}$  is concave (convex) in  $c$  and that the absolute risk aversion  $a(c)$  is decreasing and convex (constant).<sup>20</sup> Then  $\tau$  is regressive (progressive).

**Proof.** By taking the difference between (10) in two adjacent states and dividing both sides by  $y_{i+1} - y_i$ , we obtain

$$\frac{\left[ \frac{q}{u'(c_{i+1})} - \frac{q}{u'(c_i)} - \xi (a(c_{i+1}) - a(c_i)) \right]}{y_{i+1} - y_i} = \frac{\mu \left[ \frac{p'_{i+1}(e)}{p_{i+1}(e)} - \frac{p'_i(e)}{p_i(e)} \right]}{y_{i+1} - y_i}. \quad (15)$$

By assumption the right hand side increases (decreases) with  $i$ . Since the functions in the left hand side are increasing and concave (convex) in  $c_i$  and  $c_{i+1}$  - by Proposition 2 -  $c_i$  increases with  $i$ , we must have that the ratio  $\frac{c_{i+1}-c_i}{y_{i+1}-y_i} > 0$  increases (decreases) with  $i$ . **Q.E.D.**

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<sup>19</sup>That is, for each  $e > 0, i > 2$  we have

$$\frac{\frac{p'_{i+1}(e)}{p_{i+1}(e)} - \frac{p'_i(e)}{p_i(e)}}{y_{i+1} - y_i} \geq \frac{\frac{p'_i(e)}{p_i(e)} - \frac{p'_{i-1}(e)}{p_{i-1}(e)}}{y_i - y_{i-1}}.$$

<sup>20</sup>Notice that a function cannot be positive, decreasing and concave everywhere.

Notice that CARA utilities with concave likelihood ratios lead to progressive schemes. Also note, that in this case the progressivity of the payment scheme is not influenced by the presence of hidden savings since  $a(c_{i+1}) - a(c_i) = 0$ . When the likelihood ratio is convex, CRRA utilities with  $0 < \sigma \leq 1$  induce regressive schemes since  $a(c) (= \frac{c}{c})$  is strictly convex and  $\frac{1}{u'_1(c)} = c^\sigma$  is concave. Interestingly, this case includes the logarithmic utility case, which - in the observable assets case with linear likelihood ratios - would lead to *proportional* schemes. From (15) in that case we get that  $\frac{c_{i+1} - c_i}{y_{i+1} - y_i} = \mu k$  where  $k = \left( \frac{p'_{i+1}(e)}{p_{i+1}(e)} - \frac{p'_i(e)}{p_i(e)} \right) / (y_{i+1} - y_i)$ . This implies that consumption is a linear function of income. This particular case sheds light on a more general pattern under convex  $a(c)$ : the allocation with hidden savings has a more a convex relationship between output and consumption than the one with observable savings. In terms of the supporting tax/transfer system, hidden asset accumulations calls for a more regressive tax scheme. In terms of providing incentives, regressive schemes are putting more emphasis on rewards for high output levels than punishments for low output levels: More insurance for low income levels and less insurance for high income levels. Also notice that whenever  $a(c)$  is convex,  $-(a(c_i + \Delta) - a(c_i))$  is decreasing in  $i$  for any  $\Delta > 0$ . This implies that in order to keep  $-\xi(a(c_{i+1}) - a(c_i))$  constant the principal need to increase consumption more for a higher level of income (recall that  $c_{i+1} > c_i$ ). Therefore, for all HARA utilities this effect due to hidden asset accumulation always imposes an additional regressivity to the curvature dictated by  $\frac{1}{u'(c)}$ .<sup>21</sup>

**Corollary to Proposition 6** Assume FOA is valid and  $u$  is HARA. Let  $\mathbf{c}$  and  $\hat{\mathbf{c}}$  be interior optimal contracts for our model with hidden asset accumulation and the pure moral hazard model, respectively, implementing effort level  $e_0$ . If  $\hat{\mathbf{c}}$  changes with  $i$  in a convex way, than  $\mathbf{c}$  does as well.

Below we provide conditions for the validity of the FOA when  $u$  is quadratic (hence not NIARA but HARA). In order to obtain a clearer intuition of this result we examine (10) further. This expression equates discounted present value (normalized by  $p_i(e_0)$ ) of the principal's costs and benefits of increasing the agent's utility by one unit in state  $i > 1$ . This increase in utility costs the planner  $q/\beta u'(c_i)$  in consumption terms. In terms of benefits, first of all, since the participation

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<sup>21</sup>The HARA (or linear risk tolerance) class of utility functions is defined by a coefficient of absolute risk aversion

$$a(c) = \frac{1}{\delta + \gamma c},$$

with  $\delta + \gamma c \geq 0$ . This class includes CARA (for  $\gamma = 0$ ), quadratic utility (for  $\delta > 0$  and  $\gamma = -1$ ), and CRRA (for  $\delta = 0$  and  $\gamma > 0$ ). In the latter case,  $\gamma$  represents the intertemporal elasticity of substitutions, i.e., the inverse of the risk aversion parameter.

constraint is relaxed, the principal receives a return of  $\lambda$ . Further, increasing the agent's utility also relaxes the effort incentive compatibility constraint generating a return of  $\mu p'_i(e_0)/p_i(e_0)$ .<sup>22</sup> Notice that these effects are present in the standard moral hazard model as well. In the hidden asset case ( $\xi > 0$ ), there is an additional gain though: by increasing  $u(c_i)$ , the principal also alleviates the saving motives of the agent. This gain is measured (in consumption terms) by  $\xi a(c_i)$ , and it implies that the principal, *ceteris paribus*, increases consumption for every income state  $i$ . Since consumption is increasing in  $i$ , decreasing absolute risk aversion implies that these gains are getting smaller as income is getting higher, implying that the additional consumption increases due to hidden asset accumulation are decreasing. Note, however, that the difference in these gains across income levels  $i$  and  $i + 1$  determines the exact effect of hidden asset accumulation on the progressivity of the tax scheme. Under convex absolute risk aversion this difference is shrinking, hence, the consumption plan becomes more convex than in the standard moral hazard case.

## 5 Further Comparisons with the Standard Model and Extensions

In this section we analyze few other similarities and discrepancies between the two models and finally we discuss few potential extensions.

**The Value of Information** The fact that under NIARA consumption moves with the likelihood ratio alone suggests that some of the previous results about the value of information in our model satisfies similar properties to the standard case. Here is a more formal statement.

**Proposition 4** Assume NIARA, and consider an interior contract  $(\mathbf{c}, e_0)$  that is a solution of **(R)**.

Assume that the planner receives an observable and verifiable signal  $s \in S$ , and suppose that the FOA is valid in both the original and the new environment with the signal  $s$ . Then there is a new contract  $\{c_{is}\}_{i=0,N}^{s \in S}$  that strictly dominates  $\mathbf{c}$  if and only if  $s$  is *informative* about  $e_0$  in the sense of Holmström (1979).<sup>23</sup>

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<sup>22</sup>Of course, if the increase in the payment is done in a state with a negative likelihood ratio, this represents a cost since the incentive constraint is in fact tightened.

<sup>23</sup>Holmström defines an additional signal  $s$  being *informative* about the agent's action choice if and only if  $y_i$  is not a sufficient statistic for  $(y_i, s)$ , or equivalently there does not exist a function  $z$  such that

$$\hat{p}_{is}(e) = z(y_i, s) p_i(e) \text{ for all } y_i \in Y, s \in S \text{ and } e > 0.$$

**Proof (Sketch).** The proof builds on Holmström (1979). The first order condition for the new problem are

$$\frac{q}{\beta u'(c_{is})} = \lambda + \mu \frac{\hat{p}'_{is}(e_0)}{\hat{p}_{is}(e_0)} - \xi \frac{u''(c_{is})}{u'(c_{is})},$$

where - from Lemma 1 (ii) - both  $\mu > 0$  and  $\xi > 0$ . Under NIARA,  $c_{is}$  must move with  $\frac{\hat{p}'_{is}(e_0)}{\hat{p}_{is}(e_0)}$ . So, if  $s$  is informative  $c_{is}$  should move with  $s$  as well. This proves one direction of the statement. Now, assume that  $s$  is not informative. Then  $\frac{\hat{p}'_{is}(e)}{\hat{p}_{is}(e)} = \frac{p'_i(e)}{p_i(e)}$  for all  $i, e$  and  $s$ . This implies that the effort incentive constraint cannot be improved by having a consumption plan that depends on  $s$ . A reduction of uncertainty in consumption also relaxes the savings incentive constraint since NIARA implies that  $u'$  is convex. Hence:  $\sum_i p_i(e_0)u'(c_i) \geq u'(\sum_i p_i(e_0)c_i)$ . Moreover, since  $u_1$  is strictly concave and NIARA, a reduction of uncertainty allows the principal to provide the same  $U_0$  with a lower expected payment. **Q.E.D.**

We noted above that this result represents a key distinguishing feature of our model with respect to the self-insurance framework where further insurance possibilities are not available for the agent. In that case, consumption always moves with both  $i$  and  $s$  as long as  $y_{is}$  changes with them, regardless of the informational content.

**Backloading** As we emphasized in the introduction, one important discrepancy between our model and the standard moral hazard model regards the inter-temporal allocation of resources. Rogerson (1985a) has shown that, in the standard moral hazard model under additive separable preferences, the agent's optimal consumption allocation is such that the *agent is savings constrained*. This feature itself already implies that our model must induce more backloaded consumption with respect to the standard moral hazard model. We now establish an absolute result:

**Proposition 5** Whenever  $\beta \geq q$  we have  $c_0 \leq \sum_i p_i(e_0)c_i$ . The inequality is strict when, in addition,  $e_0 > 0$  and/or  $\beta > q$ .

**Proof.** The second condition in (6) implies  $qu'(c_0) \geq \beta \sum_i p_i(e_0)u'(c_i)$ . Since NIARA implies that  $u'$  is a strictly convex function of consumption, from Jensen's inequality we have  $c_0 \leq \sum_i p_i(e_0)c_i$  with strict inequality whenever  $c_i$  is not constant in  $i$ , or  $\beta > q$  (or both). We saw in the Proof of Lemma 1 (ii) that whenever  $e_0 > 0$  consumption cannot be constant in  $i$ . **Q.E.D.**

If we consider the first order conditions for the optimal contract (9) and (10) assuming  $\xi = 0$ , we recover the characteristics of the optimal contract in the standard case when asset holdings are observable and fully contractable (or to save is unfeasible to the agent). Assuming interiority of the

contract and additive separability of preferences, by multiplying (10) by  $p_i(e)$  for all  $i$  and summing across all income levels, we obtain:

$$\frac{\beta}{u'(c_0)} = q \sum_{i=1}^N p_i(e_0) \left[ \frac{1}{u'(c_i)} \right]. \quad (16)$$

Equation (16) obviously contradicts the Euler equation. Moreover, Jensen's inequality implies that, in this case, consumption actually *decreases on average* whenever  $\frac{1}{u'(\cdot)}$  is convex, which is the case for all CARA utility functions and all CRRA utilities with relative risk version parameters above 1.

**Effort Level and Insurance** Assume FOA is valid with  $e_0 > 0$ . Taking the first order conditions with respect to  $e$ , we obtain

$$q \sum_i p'_i(e_0) (y_i - c_i) + \lambda U_e(e_0, 0; \mathbf{c}) - \mu U_{ee}(e_0, 0; \mathbf{c}) + \xi U_{eb}(e_0, 0; \mathbf{c}) = 0.$$

Clearly,  $U_e(e_0, 0; \mathbf{c}) = 0$ . Moreover, it is not difficult to see that whenever the agent's problem is concave in  $e_0$  then  $-\mu U_{ee}(e_0, 0; \mathbf{c}) + \xi U_{eb}(e_0, 0; \mathbf{c}) < 0$ .<sup>24</sup> This implies that  $q \sum_i p'_i(e_0) (y_i - c_i) = q \sum_i p'_i(e_0) (y_i - c_i) + \lambda U_e(e_0, 0; \mathbf{c}) > 0$ . If we consider  $\lambda$  as an exogenous parameter defining the relative Pareto weight of the agent, this can be interpreted as an inefficiency result. Because of the informational problems, the planner implements an effort level which is lower than that dictated by production efficiency. This result is similar to the standard moral hazard case (see Rogerson, 1985b).

A natural question is whether the optimal scheme under hidden savings induces more or less consumption dispersion on the agent. We realized that this question does not have an easy answer. Intuition might suggest that since reducing consumption dispersion relaxes the savings incentives under NIARA, we might expect a reduction of consumption dispersion. In fact, in Ábrahám and Pavoni (2006), we document numerically that in the two output case ( $N = 2$ ) the infinite horizon version of the model, consumption is typically more dispersed in the hidden asset case. As similar

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<sup>24</sup>In particular, by rearranging the necessary conditions of optimality one can show the multipliers  $\mu > 0$  and  $\xi > 0$  are related as follows (details are available upon request):

$$\mu U_{eb} = \xi U_{bb}.$$

Hence, the condition  $\det H > 0$  is equivalent to

$$(U_{eb})^2 < U_{ee}U_{bb} \Leftrightarrow \mu \xi U_{bb}U_{eb} < \mu^2 U_{ee}U_{bb} \Leftrightarrow \xi U_{eb} < \mu U_{ee}.$$



result can be shown analytically here: For each *fixed* pair  $(U_0, e_0)$ , the consumption dispersion is larger in our model than in the standard case.<sup>25</sup>

## 5.1 Extensions

Before writing the concluding remarks we would like to discuss the relevance of some assumptions in the model.

**Liquidity Constrained Agents and Shallow Pockets** Suppose that the agent is liquidity constrained. Although the set of implementable levels of asset holdings  $b_0$  might be restricted by the presence of liquidity constraints, all our characterization results remain valid. First, it is obvious that given any implementable  $b_0$ , the principal is able to generate the same allocation to the agent by implementing  $b'_0 > b_0$  and adjusting transfers accordingly at no cost. Interestingly, Lemma 1 implies that whenever the FOA is valid the principal will never be able to gain by implementing a low  $b_0$  so that to make the agent liquidity constrained. In contrast, when both the agent and the planner faces liquidity constraints (or shallow pockets), it might be the case that Lemma 1 fails as  $U_b > 0$ .

**Quadratic Utility** A quite commonly used specification for  $u$  is the quadratic one:  $u(c) = -\frac{\phi}{2}(B - c)^2$  with  $\phi > 0$  and  $B \gg 0$ . This utility function does not belong to the NIARA class since  $-\frac{u''}{u'} = \frac{1}{B-c}$ , which increases with  $c$ . Most of the previous results extend to this case as well. Note in particular, that this utility function belongs to the HARA class, so Proposition 6 and relative Corollary apply in full whenever the first-order approach can be applied. The optimality conditions for problem **(R)** when  $u$  is quadratic are

$$\frac{q - \xi\phi\beta}{\beta\phi(B - c_i)} = \lambda + \mu \frac{p'_i(e_0)}{p_i(e_0)}.$$

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<sup>25</sup>Recall that  $N = 2$ . Since  $e_0$  is fixed across the two models we need to keep  $\Delta u$  constant across the two models. Since  $c_0$  is fixed as well, the discussion regarding the backloading implies that in order to satisfy the Euler equation, in our model we need to decrease  $\Gamma(e_0)u'(c_2) + (1 - \Gamma(e_0))u'(c_1)$  compared to the standard moral hazard model. The implication is that we must increase both  $c_2$  and  $c_1$  so that to keep  $\Delta u$  constant. Since  $u$  is concave,  $\Delta c$  must increase. Now, consider  $U_0$  fixed, while allowing for the possibility of changing  $c_0$ . Since as long as  $e_0$  is fixed, the principal must adjust the optimal contract so that perhaps  $c_0$  decreases, but both  $c_1$  and  $c_2$  must increase compared to the original contract, otherwise the agent would get an utility level below  $U_0$ .

It might hence be interesting to notice that since  $q - \xi\phi\beta > 0$ ,<sup>26</sup> the movements in the likelihood ratio are reflected in the optimal contract in the same way as in the standard moral hazard model. In particular, under MLR, consumption is monotone increasing in  $i$  (as in the benchmark case with NIARA utility). A sufficient condition for the validity of FOA in this case is  $f_\Gamma(e)(1 - \Gamma(e)) \leq \frac{1+q}{2}$  for  $e \geq 0$ , which is implied by  $f_\Gamma(e) \leq \frac{1+q}{2}$  since  $1 - \Gamma(e) \leq 1$ .<sup>27</sup>

**More General Conditions on  $\mathbf{p}$**  In this paper, we looked at conditions for the validity of the first order approach, which allowed simple economic interpretation. Looking at the expression of part (ii) in Lemma 2, one can notice that the key sufficient condition for the validity of the first-order approach is that the expression below is concave in  $(e, b)$ :

$$W(e, b) := \sum p_i(e) u(c_i + b).$$

An alternative route is hence to look at a new set of conditions, jointly on  $u$  and  $\mathbf{p}$  that use more heavily the shape of the optimal contract  $\mathbf{c}$  in  $i$ . So far, we have only used monotonicity (implied by NIARA and MLR). In the standard moral hazard model, Jewitt (1988) derives necessary and sufficient conditions for  $\mathbf{c}$  to change in a concave way with  $i$ . It then uses and extends results from *Total Positivity*<sup>28</sup> to find minimal conditions on  $\mathbf{p}$  so that the agent problem is concave in  $e$ . The full derivation of minimal conditions on the concavity of the agent problem jointly in  $(e, b)$  is however a task we left to future research.

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<sup>26</sup>If  $q - \xi\phi\beta < 0$ ,  $c_i$  would decrease with  $i$  when  $\frac{p'_i(e_0)}{p_i(e_0)}$  increases and vice versa. Hence:

$$\sum p_i(e_0) \frac{p'_i(e_0)}{p_i(e_0)} u(c_i) = \sum p'_i(e_0) u(c_i) < 0 \leq v'(e_0),$$

which violates the incentive compatibility constraint for  $e_0$ .

<sup>27</sup>Assume without loss of generality that  $v(e) = e$ , and for simplicity that  $\beta = q$ . After few simplifications, on the Hessian matrix

$$H := \begin{bmatrix} -\beta \sum p''_i(e) \frac{\phi}{2} (B - c_i)^2 & \beta \sum p'_i(e) \phi (B - c_i) \\ \beta \sum p'_i(e) \phi (B - c_i) & -\phi\beta [q + 1] \end{bmatrix},$$

we obtain

$$\det H = \beta^2 \phi^2 \left( \Gamma''(e) \left[ \sum \Delta\pi_i (B - c_i)^2 \right] \frac{(q+1)}{2} - (\Gamma'(e))^2 \left( \sum \Delta\pi_i (B - c_i) \right)^2 \right)$$

which implies that  $\det H > 0$  whenever

$$\frac{[-\sum \Delta\pi_i (B - c_i)^2]}{(\sum \Delta\pi_i (B - c_i))^2} \frac{(q+1)}{2} > \frac{(\Gamma'(e))^2}{-\Gamma''(e)}.$$

Finally, one can show that - since  $c_i$  and  $\Delta\pi_i$  move together (and  $\sum \Delta\pi_i = 0$ ) - we have that  $\frac{\sum \Delta\pi_i (B - c_i)^2}{(\sum \Delta\pi_i (B - c_i))^2} \geq 1$ .

**Q.E.D.**

<sup>28</sup>The key reference for Total Positivity is Karlin (1968).

## 6 Conclusions

This paper studies the two period version of the dynamic moral hazard model when agents can borrow and save on a risk-free bond market and their asset accumulation decisions are not observable. We provide sufficient conditions under which the first-order approach (FOA) is applicable in this environment. In addition to the conditions which are required in the static and in the observable savings case (MLR, CDF) we need to impose some further concavity on the problem. First of all, non-increasing absolute risk aversion (NIARA) with respect to consumption is imposed. Second, the way effort affects the probability distribution has to be concave enough or the disutility of effort has to be convex enough. This last requirement is guaranteed by the *Frisch elasticity of leisure being less than one*. One nice property of these set of sufficient conditions is that these restrictions on preferences are validated by empirical research. Another attractive aspect of them is that most popular functional forms used in applied research will satisfy these conditions. As a by-product of our analysis we identify an interesting new characterizing property of the NIARA utility functions, which might have a broader applicability.

With the help of the first order condition approach, we also characterize the optimal contract in this environment, and we mostly focus on how consumption depends on output. Similarly to the standard case, under the assumptions needed for the validity of the FOA, the optimal consumption is monotonic in output and the result by Holmström (1979) regarding the value of information in moral hazard models hold in our environment as well: The agent's consumption should vary with income if and only if the income shock is informative about the action taken by the agent. We also study extensively, the progressivity of the tax-transfer scheme supporting the optimal allocation. We identify a key force in the model which makes consumption more convex and hence the supporting tax system more *regressive* due to the possibility of hidden asset accumulation under decreasing and convex absolute risk aversion (HARA) utility functions.

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## Appendix: Proofs

**Proof of Proposition 1** The proof is based on Propositions 4 and 5 in Grossman and Hart (1983). In order to make the analogy explicit between these results and ours, we now show in detail the part of the proof related to Grossman and Hart’s Proposition 5 which regards our Corollary to Proposition 1: The sub-optimality of self-insurance. It should then be easy to see how the proof of Proposition 4 in that paper can be adapted to our environment. Further details are available upon request.

Consider a transfer scheme  $\tau$  that is optimal and it is such that  $\tau_i \equiv \tau$  is constant for all  $i > 0$ . We show that this transfer scheme cannot be optimal, by constructing an incentive compatible transfer scheme, which satisfies the agent’s participation constraint and increases the principal’s surplus. Without loss of generality, assume that  $u$  is unbounded below and recall that  $\tau_N = \tau_1$ . Consider the following modification of the scheme: leave unchanged both  $\hat{\tau}_0(\varepsilon) = \tau_0$  and  $\hat{\tau}_i(\varepsilon) = \tau$  for  $i = 2, \dots, N - 1$ , while modify  $\tau_1$  and  $\tau_N$  as follows: set  $\hat{\tau}_1(\varepsilon) = \tau + \varepsilon$ , and  $\hat{\tau}_N(\varepsilon) = \tau - \mu^\varepsilon \varepsilon$ , with  $\varepsilon > 0$ . For any  $\varepsilon$ , the value  $\mu^\varepsilon$  is chosen so that the agent is indifferent between the original plan  $\tau$  and the new one  $\hat{\tau}(\varepsilon)$ <sup>29</sup>, that is

$$\max_{e \in E, b \geq -B(\varepsilon)} u(y_0 + \tau_0 - qb) - v(e) + \beta \sum_{i=1}^N p_i(e) u(y_i + \hat{\tau}_i(\varepsilon) + b)$$

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<sup>29</sup>When  $u$  is unbounded above, we shall modify the original scheme  $\tau$  as follows:  $\hat{\tau}_0(\varepsilon) = \tau_0$ ,  $\hat{\tau}_i(\varepsilon) = \tau$  for  $i = 2, \dots, N - 1$ , and  $\hat{\tau}_1(\varepsilon) = \tau + \mu^\varepsilon \varepsilon$ , and  $\hat{\tau}_N(\varepsilon) = \tau - \varepsilon$ .

$$\begin{aligned}
&= \max_{e \in E, b \geq -B(\varepsilon)} u(y_0 + \tau_0 - qb) - v(e) + \beta \sum_{i=2}^{N-1} p_i(e) u(y_i + \tau + b) \\
&\quad + \beta [p_1(e) u(y_1 + \tau + \varepsilon + b) + p_N(e) u(y_N + \tau - \mu^\varepsilon \varepsilon + b)] \\
&= u(y_0 + \tau_0) - v(e_0) + \beta \sum_i p_i(e_0) u(y_i + \tau).
\end{aligned}$$

where for each  $\varepsilon$ ,  $B(\varepsilon) := \min_i \{y_i + \hat{\tau}_i(\varepsilon) - \underline{c}\}$ , and we stick to the particular optimal contract where  $b_0 = 0$ . Such a  $\mu^\varepsilon > 0$  exists by the Maximum Theorem. Notice indeed that both  $u$  and  $\mathbf{p}$  are continuous and by convexity and strict monotonicity,  $e$  will lie in a compact interval. Moreover, since the transfers are constant in  $i$ , and utility is unbounded below we can always choose  $\varepsilon$  in a way that  $b_0(\varepsilon) \in B(\varepsilon, \mu)$  where  $B(\varepsilon, \mu) := [\underline{c} - \min \{y_1 + \tau + \varepsilon, y_N + \tau - \mu\varepsilon\}, \frac{y_0 + \tau_0 - \underline{c}}{q}]$  is a non-empty and compact set (recall that  $\underline{c} > -\infty$ ). Finally, note that for each  $\varepsilon > 0$ , for  $\mu = 0$  the utility of the agent must be strictly larger than the one in the original contract. Since  $u$  is strictly monotone and unbounded below, and by the full support assumption, by reducing  $\mu$  we can drive the agent's utility arbitrarily low. Hence, by continuity, there must be a  $\mu^\varepsilon$  that satisfies our requirement.

We want to show that for  $\varepsilon$  small enough the difference between the principal's surplus under the new and original scheme  $\delta(\varepsilon)$  is positive. Denote by  $\hat{e}_0^\varepsilon$  and  $\hat{b}_0^\varepsilon$  the effort and asset choices of the agent under the perturbed scheme, and  $\hat{e}_0^0$ ,  $\hat{b}_0^0$  and  $\mu^0$  are the limit effort, asset and  $\mu^\varepsilon$  choices as  $\varepsilon \rightarrow 0$ . The Theorem of the Maximum implies that the optimal correspondence of choices  $\hat{e}_0^\varepsilon$  and  $\hat{b}_0^\varepsilon$  is upper hemi-continuous if considered as a function of  $\varepsilon$ . The consequence is that  $\hat{e}_0^0$  and  $\hat{b}_0^0$  are also optimal under the original scheme, hence for each  $\varepsilon > 0$  we have

$$\begin{aligned}
&u(y_0 + \tau_0 - q\hat{b}_0^0) - v(\hat{e}_0^0) + \beta \sum_i p_i(\hat{e}_0^0) u(y_i + \hat{\tau}_i(\varepsilon) + \hat{b}_0^0) \\
&\leq u(y_0 + \tau_0 - q\hat{b}_0^\varepsilon) - v(\hat{e}_0^\varepsilon) + \beta \sum_i p_i(\hat{e}_0^\varepsilon) u(y_i + \hat{\tau}_i(\varepsilon) + \hat{b}_0^\varepsilon) \\
&= u(y_0 + \tau_0 - q\hat{b}_0^0) - v(\hat{e}_0^0) + \beta \sum_i p_i(\hat{e}_0^0) u(y_i + \tau + \hat{b}_0^0).
\end{aligned}$$

Comparing the first line with the last one, we have:

$$\sum_i p_i(\hat{e}_0^0) \left[ u(y_i + \hat{\tau}_i(\varepsilon) + \hat{b}_0^0) - u(y_i + \tau + \hat{b}_0^0) \right] \leq 0. \quad (17)$$

Condition (17) can be rewritten as

$$\begin{aligned}
&\varepsilon p_1(\hat{e}_0^0) \frac{u(y_1 + \tau + \varepsilon + \hat{b}_0^0) - u(y_1 + \tau + \hat{b}_0^0)}{\varepsilon} - \\
&- \mu^\varepsilon \varepsilon p_N(\hat{e}_0^0) \frac{u(y_N + \tau + \hat{b}_0^0) - u(y_N + \tau - \mu^\varepsilon \varepsilon + \hat{b}_0^0)}{\mu^\varepsilon \varepsilon} \leq 0.
\end{aligned} \quad (18)$$

Now taking limits as  $\varepsilon \rightarrow 0$  we get

$$0 \geq p_1(\hat{e}_0^0)u'(y_1 + \tau + \hat{b}_0^0) - p_N(\hat{e}_0^0)\mu^0 u'(y_N + \tau + \hat{b}_0^0). \quad (19)$$

It is now easy to realize that  $y_N + \tau > y_1 + \tau$  and strict concavity implies  $u'(y_1 + \tau + \hat{b}_0^0) > u'(y_N + \tau + \hat{b}_0^0)$  hence it must be that  $p_N(\hat{e}_0^0)\mu^0 - p_1(\hat{e}_0^0) > 0$ .

Let us now compute the gain for the principal. For each  $\varepsilon > 0$  we have

$$\delta(\varepsilon) = p_N(\hat{e}_0^\varepsilon)\mu^\varepsilon \varepsilon - p_1(\hat{e}_0^\varepsilon)\varepsilon. \quad (20)$$

Since  $\delta(0) = 0$ , by showing  $\delta'(0) > 0$ , we show that  $\delta(\varepsilon) > 0$  for  $\varepsilon$  small enough. Note that if we divide (20) by  $\varepsilon$  and we take the limit as  $\varepsilon \rightarrow 0$  we get

$$\delta'(0) = \lim_{\varepsilon \rightarrow 0} \frac{\delta(\varepsilon)}{\varepsilon} = p_N(\hat{e}_0^0)\mu^0 - p_1(\hat{e}_0^0) > 0.$$

#### Q.E.D.

**Proof of Lemma 1** (i) We first show that the solution of the problem **(R)** must be such that (7) (the no-shirking condition) is satisfied with equality. If  $\mu > 0$  we are done. Consider the case where  $\mu = 0$ . In this case, the first-order conditions are either  $c_i = \underline{c}$  for all  $i$ , or  $\frac{q}{\beta u'(c_i)} - \xi a(c_i) = \lambda$  for all  $i$ , hence the planner fully insures the agent in period  $t = 1$  regardless of the value of  $\xi \geq 0$ . Note that full-insurance will also be good for incentive compatibility since the convexity of  $u'$  (implied by NIARA) implies  $\sum_i p_i(e_0)u'(c_i) \geq u'(\sum_i p_i(e_0)c_i)$ . That is, the planner will be able to relax the Euler equation by providing insurance. Since  $\sum_i p'_i(e_0) = 0$ , a constant  $u(c_i) = \bar{u}$  implies that  $\sum_i p'_i(e_0)\bar{u} = 0$ . Since  $v'(e_0) \geq 0$ , full insurance would imply  $v'(e_0) \geq \beta \sum_i p'_i(e_0)u(c_i) = 0$ . Combining this with (7) delivers  $v'(e_0) = \beta \sum_i p'_i(e_0)u(c_i) = 0$ .

We now show that if the solution of the problem **(R)** is interior for  $c_i$ , it must be such that the second constraint in (6) (the Euler condition) is satisfied with equality. Recall that the first-order conditions for  $c_0$  and  $c_i$  are given by (9) and (10). Again, if  $\xi > 0$  we are done. If  $\xi = 0$ , from the first order conditions we have:

$$\frac{1}{u'(c_0)} \geq \sum_i p_i(e_0) \frac{q}{\beta u'(c_i)} \geq \frac{q}{\beta \sum_i p_i(e_0)u'(c_i)}, \quad (21)$$

where the second inequality is implied by Jensen inequality. In fact, since  $1/x$  is a strictly convex transformation, this second inequality can be an equality only if the agent is fully insured. Now, comparing the first and last term in the above expression, we have  $qu'(c_0) \leq \beta \sum_i p_i(e_0)u'(c_i)$ . Combining this with (8) yields  $qu'(c_0) = \beta \sum_i p_i(e_0)u'(c_i)$ .



(ii) Whenever  $e_0 > 0$  by assumption we have  $v'(e_0) > 0$ . Therefore, full insurance is not feasible so  $c_i$  cannot be constant in  $i$ . This implies that  $\mu > 0$ . In this case we also have  $\xi > 0$ . In order to see this, recall that with  $\xi = 0$  whenever  $c_i$  is not constant in  $i$  the last inequality in (21) is satisfied with strict inequality. This however contradicts the Euler condition  $qu'(c_0) \geq \beta \sum_i p_i(e_0)u'(c_i)$  hence  $\xi$  cannot be zero either.

(iii) If  $u$  is unbounded below, since  $U_0 > -\infty$ , from the participation constraint we obviously have interiority and the previous line of proof applies. So assume  $u$  is bounded. If the Inada condition is satisfied and  $c_i = \underline{c}$  for at least one  $i > 0$ , clearly the Euler condition (8) must be satisfied. In fact, it must be that  $c_0 = \underline{c}$  as well. Since  $u$  is bounded, (7) must be satisfied with equality by the argument in (i). **Q.E.D.**

**Proof of Lemma 2** (i) We first show that under NIARA  $\sum_i p'_i(e)u'(\hat{c}_i) \leq 0$ . Note that we can rewrite the expression as  $\sum_i p_i(e) \frac{p'_i(e)}{p_i(e)}u'(\hat{c}_i)$  where  $\sum_i p_i(e) \frac{p'_i(e)}{p_i(e)} = 0$ . Moreover, from the first order condition and NIARA we have that  $\hat{c}_i$  is either constant at  $\underline{c}$  or it moves together with the likelihood ratio  $\frac{p'_i(e)}{p_i(e)}$ . Therefore, by concavity,  $u'(\hat{c}_i)$  and  $\frac{p'_i(e)}{p_i(e)}$  are negatively correlated, which proves the result.

(ii) In order to simplify the analysis, we rewrite the expression for the determinant of the Hessian as

$$\det H = \left(-v_{ee} + \beta \hat{U}_{ee}\right) \left(q^2 u_{cc} + \beta \hat{U}_{cc}\right) - \left(\beta \hat{U}_{ec}\right)^2,$$

where all terms have the obvious meaning. By the concavity of  $u$  and convexity of  $v$ , we have:  $-v_{ee} < 0$ ,  $q^2 u_{cc} < 0$ . If we now look at the expression for  $\det H$ , by making the term by term multiplications and expanding the quadratic expression, we obtain the following terms. First,  $q^2(-v_{ee})u_{cc}$  which is positive by the concavity of  $u$  and  $-v$ . Second, we have the expression  $q^2 u_{cc} \hat{U}_{ee} - \beta v_{ee} \hat{U}_{cc}$ . Both terms in the expression are positive since the concavity of  $u$  implies  $\hat{U}_{cc} < 0$  and the CDF assumption implied by the concavity of  $\Gamma$  imply that  $\hat{U}_{ee} < 0$ . The only remaining terms are  $\beta^2 \hat{U}_{ee} \hat{U}_{cc} - \beta^2 \hat{U}_{ec}^2$ , which are those in the text. **Q.E.D.**

**Proof of Lemma 3** Let  $k$  be the index so that  $\alpha_i = \pi_{ih} - \pi_{il} \geq 0$  if and only if  $i \geq k \geq 1$ . Notice that since  $\frac{\pi_{ih}}{\pi_{il}}$  increases with  $i$  and  $\sum_i \pi_{ih} = \sum_i \pi_{il} = 1$ , such  $k$  is well defined. We aim at showing that NIARA implies that

$$\frac{\sum_{i=1}^N \alpha_i u_i \sum_{i=1}^N \alpha_i u_i''}{\left(\sum_{i=1}^N \alpha_i u_i'\right)^2} = \frac{\sum_{i=2}^N \alpha_i \Delta u_i \sum_{i=2}^N \alpha_i \Delta u_i''}{\left(\sum_{i=2}^N \alpha_i \Delta u_i'\right)^2} \geq 1 \quad (22)$$

where  $\Delta u_i^{(n)} = u^{(n)}(\hat{c}_i) - u^{(n)}(\hat{c}_k)$  and we used the fact that  $\sum_{i=1}^N \alpha_i = 0 \Rightarrow \alpha_1 = -\sum_{i=2}^N \alpha_i$ . The last expression can be re-written as follows

$$\frac{\sum_{i=2}^N \alpha_i \Delta u_i \sum_{i=2}^N \alpha_i \Delta u_i''}{\left(\sum_{i=2}^N \alpha_i \Delta u_i'\right)^2} = \frac{\sum_{i=2}^N \sum_{j=2}^N \alpha_i \Delta u_i \alpha_j \Delta u_j''}{\sum_{i=2}^N \sum_{j=2}^N \alpha_i \Delta u_i' \alpha_j \Delta u_j'}.$$

Note, that if we are able to show that

$$\alpha_i \alpha_j (\Delta u_i \Delta u_j'' + \Delta u_j \Delta u_i'') \geq 2 \alpha_i \alpha_j \Delta u_i' \Delta u_j' \text{ for } \forall i, j \quad (23)$$

then we will be done. Clearly if either  $\alpha_i \alpha_j = 0$ , or  $c_i = c_k$ , or  $c_j = c_k$  the condition is satisfied since both terms are zero. So assume that  $\alpha_i \alpha_j \neq 0$ ,  $c_i \neq c_k$ , and  $c_j \neq c_k$ . Now recall that since  $\Gamma'(e) > 0$  consumption increases with  $i$ . So, we have that both  $\alpha_i \Delta u_i' < 0$  and  $\alpha_j \Delta u_j' < 0$ . We can hence divide by  $\alpha_i \alpha_j \Delta u_i' \Delta u_j'$  the inequality in (23) and it remains to show that  $\forall i, j$

$$P_{ij} \equiv \frac{\Delta u_i \Delta u_j'' + \Delta u_j \Delta u_i''}{\Delta u_i' \Delta u_j'} \geq 2. \quad (24)$$

The key aspect of our proof is to note that, if  $u$  is concave, the absolute risk aversion is decreasing if and only if the agent is more prudent than risk averse. That is, if  $-u'$  is a concave transformation of  $u$  (see Gollier, 2001, pp. 24). In other words, there is an increasing and concave function  $f$  such that

$$-u'(c) = f(u(c)) \text{ for all } c.$$

Now, note that

$$u''(c) = \frac{du'(c)}{dc} = -\frac{d(-u'(c))}{dc} = -f'(u(c))u'(c) = f'(u(c))f(u(c)). \quad (25)$$

If we set  $u(c_i) = z_i$ , (24) becomes

$$P_{ij} = \frac{\Delta z_i \Delta (f_j f_j') + \Delta z_j \Delta (f_i f_i')}{\Delta f_i \Delta f_j} = \frac{(z_i - z_k)(f'(z_j)f(z_j) - f'(z_k)f(z_k)) + (z_j - z_k)(f'(z_i)f(z_i) - f'(z_k)f(z_k))}{(f(z_i) - f(z_k))(f(z_j) - f(z_k))}. \quad (26)$$

If  $z_l > z_k$  (resp.  $z_l < z_k$ ), from the Mean value theorem, for  $\forall l \exists z_l^* \in [z_k, z_l]$  (resp.  $z_l^* \in [z_l, z_k]$ ) such that

$$f(z_l) - f(z_k) = f'(z_l^*)(z_l - z_k). \quad (27)$$

Substituting (27) into (26) we obtain

$$P_{ij} = \frac{f'(z_j)f(z_j) - f'(z_k)f(z_k)}{f'(z_i^*)(f(z_j) - f(z_k))} + \frac{f'(z_i)f(z_i) - f'(z_k)f(z_k)}{f'(z_j^*)(f(z_i) - f(z_k))}.$$

Some algebra gives us

$$P_{ij} = \frac{f'(z_j^*)}{f'(z_i^*)} \frac{\frac{f'(z_j)}{f'(z_j^*)}f(z_j) - \frac{f'(z_k)}{f'(z_j^*)}f(z_k)}{f(z_j) - f(z_k)} + \frac{f'(z_i^*)}{f'(z_j^*)} \frac{\frac{f'(z_i)}{f'(z_i^*)}f(z_i) - \frac{f'(z_k)}{f'(z_i^*)}f(z_k)}{f(z_i) - f(z_k)}.$$

Now the required result is given by the following two inequalities:

$$P_{ij} \geq \frac{f'(z_j^*)}{f'(z_i^*)} + \frac{f'(z_i^*)}{f'(z_j^*)} \geq 2. \quad (28)$$

In order to obtain the first inequality, notice the following facts for both  $l = i, j$ . First, since  $f$  is concave we have that whenever  $f(z_l) - f(z_k) > 0$  (resp.  $f(z_l) - f(z_k) < 0$ ) then  $f'(z_l) \leq f'(z_l^*) \leq f'(z_k)$  (resp.  $f'(z_l) \geq f'(z_l^*) \geq f'(z_k)$ ) implying that  $0 \leq \frac{f'(z_l)}{f'(z_l^*)} \leq 1$  and  $\frac{f'(z_k)}{f'(z_l^*)} \geq 1$  (resp.  $\frac{f'(z_l)}{f'(z_l^*)} \geq 1$  and  $\frac{f'(z_k)}{f'(z_l^*)} \leq 1$ ). This together with the fact that  $f$  is negative leads to

$$\frac{\frac{f'(z_l)}{f'(z_l^*)}f(z_l) - \frac{f'(z_k)}{f'(z_l^*)}f(z_k)}{f(z_l) - f(z_k)} \geq 1$$

for  $l = i, j$ . This gives the first inequality in (28). The second inequality is implied by the simple mathematical fact that for all  $a > 0$  we have  $\frac{1}{a} + a \geq 2$ , applied to  $a = \frac{f'(z_j^*)}{f'(z_i^*)}$ .<sup>30</sup> This completes the proof, since  $P_{ij} \geq 2$  for  $\forall i, j \geq 1$  implies  $R_u \geq 1$ .

The proof of the sufficiency part is very easy. Since a concave  $u$  is NIARA *if and only if*  $-u'$  is more concave than  $u$ , if  $u$  is not NIARA we can easily find a counterexample with some monotone weights and (weakly) monotone consumption allocation where  $P_{ij} < 2$  and therefore  $R_u < 1$ . Details are available upon request. **Q.E.D.**

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<sup>30</sup>Since  $(a - 1)^2 = a^2 - 2a + 1 \geq 0$ , dividing both sides by  $a > 0$  one obtains  $a + \frac{1}{a} - 2 \geq 0$  as desired.